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## **Loans, ordering and shortage costs in start-ups: a dynamic stochastic decision approach**

Edgar Possani<sup>1</sup>, Lyn C. Thomas<sup>1</sup>, Thomas W. Archibald<sup>2</sup>

<sup>1</sup> *University of Southampton, UK; and* <sup>2</sup> *University of Edinburgh, UK*

### **Abstract**

Start-up companies are a vital ingredient in the success of a globalised networked world economy. We believe such companies are interested in maximising the chance of surviving in the long-term. We present a Markov decision model to analyse survival probabilities of start-up manufacturing companies. Our model examines the implications of their operating decisions in particular their inventory strategy to include purchasing, shortage and ordering costs, as well as the influence of loans on the firm. It is shown that although the start-up company should be more conservative in its component purchasing strategy than if it were a well established company it should not be too conservative. Nor is its strategy monotone in the amount of capital available.

### **Keywords:**

Inventory control, dynamic programming, risk analysis, manufacturing.

### **Introduction**

Start-up companies are a vital ingredient in the success of a nation's economy and yet there is a high failure rate for such companies. Thus it is important to identify strategies that ensure success for these newly formed, capital constrained companies. These are not necessarily the same strategies that are best for well established companies with access to large, if not infinite, amounts of capital. The main difference is that start up companies are primarily interested in the probability of survival as opposed to maximising the average profit, which is the criterion found in almost all previous management science models and is the one best suited to well established companies.

This paper looks at a simplified inventory problem which typifies the problems of many start-up manufacturing companies (and those where the main profit is through adding value to bought in components). This builds on the inventory model introduced by Archibald *et al*<sup>1</sup> by allowing ordering and transportation costs, shortage effects and

the ability for the firm to secure overdraft facilities based on its inventory and work in stock. The objective is to find the optimal inventory strategies for small firms, see how they vary as the capital available rises, and contrast the strategies with the profit maximising ones, which large firms should employ.

There is little work done on making joint production and financial decisions for (start-up) companies. Traditionally, finance and production decisions have been taken separately. This is largely due to the Modigliani-Miller<sup>2</sup> proposition, that the decisions could be taken independently under perfect and competitive markets. However, in reality markets are imperfect (there are tax advantages of debt, bankruptcy cost, etc.), and as Cifarelli *et al*<sup>3</sup> mention, it has led to overlooking some of the dynamic implications of the structure of capital. They study stochastic models to analyse the value of a firm over time, under different financing policies, and are interested in the probability of bankruptcy and repayment of loans. However, their analysis is not for start-up companies, and they do not consider the influence the ordering policy has on the probability of survival of the firm.

Freel<sup>4</sup> discusses the way product innovating small manufacturing firms within the UK obtain their finance and point out the limited capital available to such firms. They suggest that such companies need to watch this cash flows carefully in order to survive. There are very few papers so far that analyse the relationship between capital available and production decisions in start up companies. Buzacott and Zhang<sup>5</sup> analyse a periodic review inventory problem with deterministic demand under different combinations of secured and unsecured loans and original capital. They do not explicitly consider the probability of survival as an objective whereas Archibald *et al*<sup>1</sup> do so for a simpler inventory model than the one considered here.

In the following section, we present the survival probability model we analyse for a start-up company and its properties. We show how it behaves in similar ways to Archibald *et al*<sup>1</sup> model, highlighting the behaviours that concern the loans, ordering and shortage costs. In the next section we present some numerical examples with changes in these variables. Afterwards we analyse the average reward model for a mature company, and prove that the optimal inventory policies are more cautious for the survival model than for the profit maximising model. Our conclusions are summarised in the last section, where we outline possible further work.

# Survival Probability Model

The simplified model we study is as follows. The manufacturer makes one type of unit from a single type of components that has to be purchased from other manufacturers. We consider the lead time for ordering as fixed, and taken as 1 time period. The demand for the unit is random with independent identical distributions each time period. If the demand in a period exceeds the number of components available in stock then the excess demand is lost at a shortage cost of  $r$ . The shortage cost we consider is not a stockout cost (since we will deal with capital flows explicitly), but rather the administrative cost involved in not supplying the demand. The decision the firm faces is how many components to order each period.

Figure 1 shows a time line for the events in one period. We assume that the bank checks the finances of the firm at the end of each period. We pay the supplier on receiving the components, and not on ordering. The components are delivered at the end of each period, having been ordered at the beginning of the period. We assume units cannot be manufactured if components are not available. These assumptions really mean that the manufacturing time and any variation in lead time is small compared with the lead time itself. Ordering too many components ties up the firm's capital in stock which is not required, whereas ordering too few leads to unsatisfied demand and hence a loss of profit.

Let  $S$  be the selling price of each unit;

$C$  be the cost of buying a single component;

$H$  be fixed overhead costs per period (e.g. cost of staff and premises) which are incurred irrespective of the activity of the firm;

$w$  be the cost of submitting an order for components including fixed transportation costs;

$r$  be the shortage cost per unit of unsatisfied demand;

$p(d)$  the probability of there being a demand of  $d$  units;

$M = \max\{d | p(d) > 0\}$  be the maximum possible demand that can be satisfied in a period (this can be interpreted as the maximum production capacity in a period), and

$\bar{d} = \sum_{d=0}^M dp(d)$  be the average demand per period.

We define  $q(n, i, x)$  as the maximum probability that a firm will survive  $n$  more periods given that it has  $i$  components in stock and  $x$  units of capital. We assume that storage constraints put some upper limit on  $i$  and thus we have a finite action space since we cannot order more than this amount.  $q(n, i, x)$  is the optimal function for a finite horizon dynamic programming problem with a countable state space ( $x$  assumed to have discrete levels), and a finite action space - the amount  $k$  to order. Thus it has an optimal non-stationary policy, see Puterman<sup>6</sup>. Moreover, the survival probability  $q(n, i, x)$  satisfies the following dynamic programming optimality equation

$$q(n, i, x) = \max_k \left\{ \sum_{d=0}^M p(d) q(n-1, i+k-\min(i, d), x + S \min(i, d) - Ck - r \max(0, d-i) - \min(w, kw) - H) \right\}. \quad (1)$$

As extra cash is available through the borrowing on the inventory, the bankruptcy line for this model differs from the one presented by Archibald *et al*<sup>1</sup>. The loan on the assets (inventory) is made explicit in the boundary conditions which is  $q(n, i, x) = 0$ , if  $x + \alpha Ci < 0$ , and  $q(0, i, x) = 1$  for  $i \geq 0$ , and  $x + \alpha Ci \geq 0$ , where  $\alpha$  is the proportion of the collateral that the bank uses in setting the loan. The loan we are considering is more of a variable overdraft, as no interest rate is charged for it in equation (1). Note that  $0 \leq \alpha < 1$ . Recall that  $C$  was the purchasing price of one unit of component, thus  $Ci$  is the value of the inventory on stock, and  $\alpha Ci$  is the valuation by the bank of the inventory  $i$  on stock, that is, the loan the firm would receive at the beginning of the period. If the firm goes bankrupt (i.e. if at a given period  $x + \alpha Ci < 0$ ) any inventory in stock is taken by the bank.

Note that  $\min(w, kw)$  is the cost of making an order. When the order quantity is  $k > 0$ ,  $\min(w, kw) = w$  is the cost of ordering components, and when no components are ordered,  $k = 0$ ,  $\min(w, kw) = 0$ , there is no ordering cost. For each unit of unsatisfied demand  $\max(0, d-i)$  there is a cost of  $r$ , thus  $r \max(0, d-i)$  is the shortage cost. If the demand is smaller than the inventory available then there is no shortage cost, however, if the demand in a given period is higher than can be supplied from the inventory in stock ( $d > i$ ) then we incur a cost of  $r(d-i)$ . If there are no shortage costs,  $r = 0$ , no ordering costs,  $w = 0$ , and no loans  $\alpha = 0$ , we are left with the original model studied by Archibald *et al*<sup>1</sup>.

We denote by  $k(n, i, x)$  the optimal action in state  $(n, i, x)$  for given inventory level  $i$  and capital  $x$  where  $k(n, i, x) = \operatorname{argmax}\{\sum_{d=0}^M p(d) q(n-1, i+k-\min(i, d), x + S \min(i, d) - Ck - r \max(0, d-i) - \min(w, kw) - H)\}$ .

## *Properties*

There are some obvious properties which one expects of the survival probability  $q(n, i, x)$  as  $n, i, x$  varies, that are similiar to the ones obtained in Archibald *et al*<sup>1</sup>. We also expect the probability of survival not to increase with higher shortage and ordering cost, or with lower valuations of the stock from the bank. These properties are summarised in Lemma 1, and Theorem 1, where the proofs of all lemmas are given in the Appendix.

### **Lemma 1**

- i)  $q(n, i, x)$  is non-increasing in  $n$ .
- ii)  $q(n, i, x)$  is non-decreasing in  $x$ .
- iii)  $q(n, i, x)$  is non-decreasing in  $i$ .

### **Theorem 1**

- i)  $q(n, i, x)$  is non-decreasing in  $\alpha$ .
- ii)  $q(n, i, x)$  is non-decreasing in  $S$ .
- iii)  $q(n, i, x)$  is non-increasing in  $C$ .
- iv)  $q(n, i, x)$  is non-increasing in  $H$ .
- v)  $q(n, i, x)$  is non-increasing in  $r$ .
- vi)  $q(n, i, x)$  is non-increasing in  $w$ .

### **Proof:**

i) The proof is by induction on  $n$ . Since  $q(n, i, x) = 0$  when  $x + \alpha Ci < 0$  for all  $n$ ,  $q(0, i, x) = 1$  when  $x + \alpha Ci \geq 0$  for  $i \geq 0$  and  $q(1, i, x) \leq 1$  when  $x + \alpha Ci \geq 0$  and  $i \geq 0$ , the hypothesis holds in the case  $n = 0$ . Assume the hypothesis holds for  $n$ , and that  $\alpha_1 \leq \alpha_2$  while values  $r, w \geq 0$  are fixed. Let  $q'(n, i, x, \alpha)$  be the survival probability when  $\alpha$  is the collateral fraction assumed by the bank. Use  $\max_i\{a_i\} - \max_i\{b_i\} \leq \max_i\{a_i - b_i\}$  to show that:

$$q'(n+1, i, x, \alpha_1) - q'(n+1, i, x, \alpha_2) \leq \max_k \left\{ \sum_{d=0}^M p(d) q'(n, i+k-\min(i, d), x + S \min(i, d) \right.$$

$$\begin{aligned}
& -r \max(0, d - i) - Ck - \min(w, kw) - H, \alpha_1) \\
& -q'(n, i + k - \min(i, d), x + S \min(i, d) \\
& -r \max(0, d - i) - Ck - \min(w, kw) - H, \alpha_2) \} \leq 0
\end{aligned}$$

Hence, hypothesis (i) holds for  $n + 1$ .

ii) Note that in equation (1) there is more capital available with a sale price  $S_1$  than with a sale price  $S_2$  when  $S_1 > S_2$ , hence by Lemma 1 ii), the property is valid.

iii)- vi) Note that in equation (1) there is less capital available under component cost  $C_1$  than component cost  $C_2$  when  $C_1 > C_2$ , and hence by Lemma 1 ii), the property (iii) is valid. We can use the same argument for the overhead costs, shortage costs and ordering costs separately to justify properties iv), v), and vi), as they have the same effect in capital.

□

Lemma 1, and Theorem 1 confirm that a firm is better off with more inventory, capital, and higher loans, or with smaller ordering, and shortage costs. On the other hand the probability of surviving cannot increase as the time horizon increases.

It is of interest to know for what values of  $x$  there is no chance of survival and for what  $x$  the survival is certain. If  $p(0) > 0$  one extreme is a continual zero demand for the product. Even in this case the firm can survive if its initial capital is enough to cover its costs in each period. Then, under a continual zero demand  $q(n, i, x) = 1$  if  $x + \alpha Ci > Hn$ . However, at the other extreme if the demand goes up to  $M$ , and we have no stock in inventory, then we will also have to pay the shortage cost  $rM$ . Hence in general  $q(n, i, x) = 1$  if  $x > (H + rM)n - \alpha Ci$ .

Looking at the situation when we cannot survive, we should investigate the most advantageous run of demands. To avoid shortage costs the demand should not be bigger than the inventory in stock, and so the best we can hope for is to have a demand of exactly the same amount of the items in stock. Each period the firm only makes a profit, and hence improves its financial position if the amount ordered is at least  $k^* = \lfloor H + w/(S - C) \rfloor + 1$  (so that  $(S - C)k^* > H + w$ , the profit is bigger than the costs) where  $\lfloor y \rfloor$  denotes the integer part of  $y$ . Thus, if we start with zero components in stock, we can only hope to survive if we can order  $k^*$  or more components initially. That is,  $x + \alpha Ck^* \geq Ck^* + H + w$  where  $\alpha Ck^*$  is the loan we get on inventory at the end of the first period. Thus, if  $x < (1 - \alpha)Ck^* + H + w$  there is no chance of survival.

If we start with  $0 < i < k^*$  components in stock, then the only difference is that the money available to us at the start is  $x + \alpha Ci$ . If we sell all  $i$  items at the end of the period we have a capital of  $x + Si - Ck^* - H - w$ , and can borrow  $\alpha Ck^*$ . Hence, we will not survive if  $x < (1 - \alpha)Ck^* + H + w - Si$ .

Let us now focus on the properties that relate the capital available with the stock in inventory.

**Lemma 2**

- i) For any  $i \geq M$ ,  $q(n, i, x) \geq q(n, i + j, x - Cj - w) \forall j \geq 0$ , and  $x + \alpha Ci \geq 0$ .
- ii) For  $i < M$   $q(n, i + j, x) \leq q(n, i, x + (S + r)j)$ , and for  $i \geq M$ ,  $q(n, i + j, x) \leq q(n, i, x + Sj) \forall j \geq 0$ , and  $x + \alpha Ci \geq 0$ .

The proof is given in the Appendix.

In other words, when  $i \geq M$ , we prefer to have  $Cj + w$  capital instead of  $j$  components in stock, on the other hand for an inventory level  $i > M$  we prefer to have  $Sj$  extra cash in capital than  $j$  units in stock, and for  $i \leq M$  we prefer  $Sj + r$  extra capital than  $j$  items in stock.

We now consider the probability of the firm surviving over an infinite horizon.  $q(i, x) = \lim_{n \rightarrow \infty} q(n, i, x)$  is the probability that the firm will survive forever given that it has  $i$  components in stock and  $x$  units of capital. We describe the properties of the function  $q(i, x)$  in the Lemma 3 and Theorem 2.

**Lemma 3**

- i)  $q(i, x) = \lim_{n \rightarrow \infty} q(n, i, x)$  exists.
- ii) For any  $i \geq M$ ,  $q(i, x) \geq q(j + i, x - jC) \forall j \geq 0$ ,  $x + \alpha Ci \geq 0$ .
- iii) For  $i > M$ ,  $q(i + j, x) \leq q(i, x + jS)$ , and for  $i \leq M$ 

$$q(i + j, x) \leq q(i, x + jS + r) \forall j \geq 0, x + \alpha Ci \geq 0.$$
- iv)  $q(i, x)$  is non-decreasing in  $x$ .
- v)  $q(i, x)$  is non-decreasing in  $i$ .

The proof of the lemma is given in the Appendix.

**Theorem 2**

- i)  $q(i, x)$  is non-decreasing in  $\alpha$ .
- ii)  $q(i, x)$  is non-decreasing in  $S$ .



- iii)  $q(i, x)$  is non-increasing in  $C$ .
- iv)  $q(i, x)$  is non-increasing in  $H$ .
- v)  $q(i, x)$  is non-increasing in  $r$ .
- vi)  $q(i, x)$  is non-increasing in  $w$ .

**Proof:**

From Lemma 3 we know that the  $q(i, x)$  exists. Hence, points i)-vi) follow immediately by taking the limit in the results of Theorem 1 i)-vi).

□

However we still have to prove that these results are non-trivial, namely that there are levels of capital  $x$  where survival is a real possibility, i.e  $q(i, x) \neq 0$ .

**Theorem 3** *For all inventory levels  $i$ ,  $q(i, x) > 0$  for some finite  $x$ .*

**Proof:**

Consider  $\alpha = 0$  (that there is no loan), and a policy that orders up to  $2M$  each period. Such a policy does not incur any shortage cost, except perhaps in the first period. Hence, Theorem 1 of Archibald *et al*<sup>1</sup> shows that with an initial capital of  $2MC + H + w + rM$  there is a positive probability of survival if we take the overhead costs for the reduced model to be  $H' = H + w$  (note that having an overhead cost of  $H$  and an ordering cost of  $w$  ensures that the firm has at least as much capital as when the overhead cost is  $H'$ ). Hence, the result is true for  $\alpha = 0$ .

On the other hand if  $\alpha > 0$ , because of Theorem 2 i) the result also holds, as more capital is available through the loan.

□

## Experimental Results

We present a set of comparisons for the behaviour of the model under different values of  $\alpha$ , the ordering cost  $w$ , and the inventory level  $i$ . In the example the holding cost is  $H = 10$ , the cost of one component is  $C = 3$ , and the selling price of each item produced is  $S = 5$ , and we have a Poisson demand process with mean 7.5 truncated at 20 (i.e the probability that the demand is higher than 20 is added to the probability

of the demand equals 20). These relate to the prices and costs found by an Australian air conditioning unit manufacturer where the main component is the generator.

First we analyse the effect different collateral levels  $\alpha$  have on the probability of survival and the ordering policy. We consider a case where the ordering and shortage costs are zero. In Figure 2 we present, for a given inventory level  $i = 3$ , a graph comparing the probability of survival as capital increases, and a graph comparing the ordering policy with different levels of capital.

Note in the first graph of Figure 2 how the probability of survival increases with bigger loans (better  $\alpha$  valuation by the bank of the inventory in stock). This is what we expected (recall Theorem 1 i)). With a capital of  $x = 13$  and no loan ( $\alpha = 0$ ) the survival probability is 0.34, whereas with  $\alpha = 0.3$  the probability of survival is 0.74, and with  $\alpha = 0.8$  it is 0.99. We need less capital to start surviving when the collateral level  $\alpha$  is higher. Hence, for a capital of  $x = 3$  there is a probability of 0.95 of surviving with  $\alpha = 0.8$ , but there is no chance of surviving in the long run for  $\alpha = 0.3$ . This confirms Buzacott and Zhang<sup>4</sup> who found that the bank's assessment of the risk associated with the firms inventory had a large impact on its profitability. In our case it has a large effect on the probability of survival in the long run.

Archibald *et al*<sup>1</sup> found that as the capital level increased we stopped having a unique policy - single  $k$  policy - and we started to have multiple ordering policies - several values for  $k$  - that maximised the survival probability. A parsimonious policy is one that orders the least amount possible - the smallest of such  $k$  values. We show in the second graph of Figure 2 the parsimonious policy (and the point  $T$  where we start having multiple policies). Note in the second graph of Figure 2 how the firm cannot be too conservative in its ordering policy, and how we need for small quantities of capital to order at least  $k^* = 6$  components to survive (i.e for  $q > 0$ ), for any  $\alpha$  level. This graph also shows that the order policy is non-monotonic with respect to capital. For example, for  $\alpha = 0.3$  and capital  $x = 13$  the unique optimal ordering policy is  $k = 8$ ; however for  $x = 14$ , the unique optimal ordering policy is  $k = 7$ . This is because of short term effects of what size of demand next period guarantees survival for the next two periods at least. It also seems that for bigger loans the firm is willing to buy more components; the parsimonious policy for  $\alpha = 0$  buys at most 13 items, whereas for  $\alpha = 0.8$  the parsimonious policy buys at most 16 items. However, this is not true for any given capital (and inventory) level; for  $x = 24$  with  $\alpha = 0$  the optimal

policy is  $k = 8$  whereas with  $\alpha = 0.3$  the optimal policy is  $k = 7$ . We cannot state that the ordering policy will always be bigger for bigger  $\alpha$  levels (when  $x$ , and  $i$  are fixed). Again this is caused by a short term effect of trying to maximise the demand next period which ensures survival over the next two periods. On the other hand, the ordering policy is not more than  $k = 16$  components, this is reasonable as we have  $i = 3$  components in stock, and the maximum demand possible in one period is 20, and there is no shortage cost.

In Figure 3 we show the behaviour of the probability of survival and the ordering costs for different inventory levels  $i = 3, 5$ , and 7. In this case we are considering a shortage cost of  $r = 0.5$ , and a loan of  $\alpha = 0.5$ . In the first graph of Figure 3 we can see how the probability of survival increases with larger inventory levels. This is also what happened when the collateral for the loan was bigger. For higher inventory levels of stock the ordering policy tends to be smaller. Note how for  $i = 3$  we do not order more than  $k = 22$  components (ordering exactly that amount for  $x = 62, 64$ ), where as for  $i = 5$  we do not order more than  $k = 13$  components. However, we cannot say that the ordering policy will always be smaller for bigger  $i$  inventory levels (when  $x$ , and  $\alpha$  are fixed). For example two cases where the policy for  $i = 3$  is smaller than for  $i = 5$  and  $i = 7$  are: 1) for capital  $x = 9$ , the optimal policy for  $i = 3$  is  $k = 7$ , but for  $i = 5$  it is  $k = 9$ , and 2) for capital  $x = 4$ , the optimal policy for  $i = 3$  is  $k = 6$ , but for  $i = 7$  it is  $k = 7$ . Note how the ordering policy for  $i = 3$  can be bigger than 17 items, which did not happen before (second graph Figure 2) when there were no shortage cost. In this case as there is a shortage cost we expect the firm to order more to cover a possible high demand in a given period.

To analyse the effect of different ordering costs, we have tried  $w = 0, 3, 30$  with two different capital levels  $x = 30, 50$ . In Figure 4, we present a graph showing the changes in the survival probability and optimal policy as the inventory increases. In the first graph we have not included the results for  $w = 0, x = 30$ , as it is very similar to  $w = 3, x = 50$ , and the results for  $w = 0, x = 30$ , which gave a line that almost went in  $q = 1$  across for all  $i$  inventory levels. Note how the probability of survival increases as the inventory level increases. It is clear that a smaller ordering cost produces a higher probability of survival. In the second graph of Figure 4 we show the ordering policy as the inventory increases. Note how for very high ordering costs  $w = 30$  (three times the overheads) the company will try not to order, and so it needs high initial inventory

levels to survive (e.g.  $i > 30$ ). The company will also try to order more components when  $w = 3$  than when no ordering costs are involved (i.e  $w = 0$ ). Similar results occur for shortage costs.

## Average Reward Model

In this section we analyse the model for an established manufacturing firm. We assume that the objective for such a company is to maximise the average reward (profit) per period. In this case there is no constraint on the amount of capital as it is assumed that the firm has enough to finance any purchase it wants. Thus, the state of a firm at the start of any period is described completely by the number of components in stock. This is a countable state, finite action, unichained Markov decision process. Hence, the standard results for the average reward Markov decision processes hold (see Puterman<sup>6</sup>).

Let  $g$  be the average reward per period under the inventory policy that optimises the average reward period and let  $v(i)$  be the bias term of starting with  $i$  items in stock. The optimality equation of the dynamic programming model for the firm under the given assumptions is

$$g + v(i) = \max_k \left\{ \sum_{d=0}^M p(d) (S \min(i, d) - kC - r \max(0, d - i) - \min(w, kw) - H + v(i + k - \min(i, d))) \right\} \quad (2)$$

Let  $k(i) = \operatorname{argmax}_k \{ \sum_{d=0}^M p(d) (S \min(i, d) - kC - H - v(i + k - \min(i, d))) \}$  be the optimal policy for a given inventory level  $i$ .

We assume there is a limit  $N$  on the components the firm may buy in any given period, which represents the constraint on the available supply on the market.  $N$  can be considered much larger than  $M$  (at least  $2M < N$ ). Recall that a parsimonious policy is one that orders the least amount possible to maximise the survival probability. Our aim is to compare the parsimonious policy for the average reward model with that of the survival probability introduced in the second section.

**Lemma 4** *The optimal average reward and bias terms that satisfy the model stated in equation (2) are*

$$g = (S - C)\bar{d} - H - \frac{w\bar{d}}{N},$$

$$v(i) = \begin{cases} Ci + (S - C)\bar{d} + \frac{iw}{N} & i \geq M, \\ Si - (S - C) \sum_{d=0}^i (i - d)p(d) - r \sum_{d=i+1}^M (d - i)p(d) & i < M, \\ \quad + \sum_{d=0}^i p(d)(i - d)\frac{w}{N} & \end{cases} \quad (3)$$

where the parsimonious policy is given for  $w > 0$  by

$$k(i) = \begin{cases} 0 & \text{if } i \geq 2M \\ N & \text{if } i < 2M, \end{cases} \quad (4)$$

and for  $w = 0$  by

$$k(i) = \begin{cases} 0 & \text{if } i \geq 2M \\ 2M - i & \text{if } M < i < 2M \\ M & \text{if } i \leq M. \end{cases} \quad (5)$$

The proof is given in the Appendix.

Note that the policy when  $w > 0$  suggest that the company should buy as many components as are available in the market if the inventory level is smaller than  $2M$ . The firm is trying to avoid the payment of ordering costs as much as possible. On the other hand when there is no ordering cost  $w = 0$  we will only order up to  $M$  components if the inventory level is less than  $M$ , and no more than  $2M - i$  if the inventory level is bigger than  $M$ . As there is no ordering cost the company orders whenever it is needed, depending on the inventory in stock, and not more than  $2M$  components each period.

We now compare both the profit maximising model and the survival probability one. We show that a company should be more cautious and risk averse in its ‘survival’ phase than in its mature ‘profit maximising’ phase. Caution means ordering fewer components. Buying a large number of components depletes the capital reserves, but gives the chance of high profits if the demand is high (as well as reduce the shortage cost). On the other hand, buying fewer items depletes the reserves less, but one gives up the chance of higher profits in case the demand is high in the next period. The following theorem shows that the parsimonious policy for the survival probability function is more cautious than the parsimonious policy for the average reward if  $w > 0$ . Let  $k'(n, i, x)$  and  $k'(i, x)$  be the parsimonious policy for the survival probability model in the finite, and infinite horizon respectively. That is the smallest optimal  $k \in k(n, i, x)$ ,

and  $k \in k(i, x)$  (i.e.  $k'(n, i, x) = \min\{k(n, i, x)\}$ ,  $k'(i, x) = \min\{k(i, x)\}$ ). Likewise let  $k'(i)$  be the parsimonious policy for the average reward model.

**Theorem 4** *If  $w > 0$  then*

*i)  $k'(n, i, x) \leq k'(i)$  for all  $n, i$ , and  $x$ .*

*ii)  $k'(i, x) \leq k'(i)$  for all  $i$ , and  $x$ .*

**Proof**

*i)* Since for  $i \leq 2M$ ,  $k'(i)$  is  $N$  the largest possible order, then trivially  $k'(n, i, x) \leq k'(i)$ , and  $k'(i, x) \leq k'(i)$ .

Define

$$Q(k, n, i, x) = \left\{ \sum_{d=0}^M p(d) q(n-1, i+k-\min(i, d), x+S\min(i, d)-Ck -r \max(0, d-i) - \min(w, kw) - H) \right\}.$$

For  $i \geq 2M$ , we need to show that in the survival probability case, the optimal order is zero. Suppose we take an order of size  $\delta$ , then

$$\begin{aligned} Q(\delta, n, i, x) &= \sum_{d=0}^M p(d) q(n-1, i+\delta-d, x+Sd-C\delta-w-H) \\ &\leq \sum_{d=0}^M p(d) q(n-1, i-d, x+Sd-w-H) \\ &\leq \sum_{d=0}^M p(d) q(n-1, i-d, x+Sd-H) = Q(0, n, i, x) \end{aligned}$$

where the first inequality holds from Lemma 2 i), and the second from Lemma 1 ii).

Hence  $k(n, i, x) = 0 \leq k(i)$ .

*ii)* The proof that  $k(i, x) \leq k(i)$  follows in exactly the same way using Lemma 3 instead of Lemma 2.

□

The results also hold in the case  $w = 0$ , from the proofs in Archibald *et al*<sup>1</sup>. Hence,

**Theorem 5**

*If  $w=0$ , then*

*i)  $k'(n, i, x) \leq k'(i)$  for all  $n, i$ , and  $x$ .*

*ii)  $k'(i, x) \leq k'(i)$  for all  $i$ , and  $x$ .*

## Conclusion

We have presented three different extensions to the model presented by Archibald *et al*<sup>1</sup> and highlighted their validity in different ways. We showed the differences in the inventory policies between a well established firm and a start-up company with limited capital to include loans, ordering, shortage and transportation costs in our analysis.

We have shown that to maximise the probability of survival start-up companies should employ more conservative strategies for ordering components parts than more established firms. We have shown the influence the different costs have on the probability of survival and in the ordering policy. These include how having as big as possible collateral for the loans increases the probability of survival, and how the shortage and ordering cost translates into bigger orders, and smaller probabilities of survival.

It will also be of interest to consider further extension to the model to include marketing and technology decisions, as well as the case of several components and products.

# Appendix

## Proof of Lemma 1

The proof is by induction on  $n$ . Since  $q(n, i, x) = 0$  when  $x + \alpha Ci < 0$  for all  $n$ ,  $q(0, i, x) = 1$  when  $x + \alpha Ci \geq 0$  for  $i \geq 0$  and  $q(1, i, x) \leq 1$  when  $x + \alpha Ci \geq 0$  and  $i \geq 0$ , all three hypotheses hold in the case  $n = 0$ .

Assume all three hypotheses hold for  $n$ , and use  $\max_i \{a_i\} - \max_i \{b_i\} \leq \max_i \{a_i - b_i\}$  to show for given values of  $r, w \geq 0$ , and  $\alpha \geq 0$  that

$$\begin{aligned} \text{i) } q(n+1, i, x) - q(n, i, x) &\leq \\ &\max_k \left\{ \sum_{d=0}^M p(d) \left( q(n, i+k - \min(i, d), x + S \min(i, d) - kC \right. \right. \\ &\quad \left. \left. - r \max(0, d-i) - \min(w, kw) - H \right) \right. \\ &\quad \left. - q(n-1, i+k - \min(i, d), x + S \min(i, d) - kC \right. \\ &\quad \left. - r \max(0, d-i) - \min(w, kw) - H \right) \} \leq 0. \end{aligned}$$

Hence, hypothesis (i) holds for  $n+1$ .

$$\begin{aligned} \text{ii) } q(n+1, i, x) - q(n+1, i, x+a) &\leq \\ &\max_k \left\{ \sum_{d=0}^M p(d) \left( q(n, i+k - \min(i, d), x + S \min(i, d) - kC - r \max(0, d-i) \right. \right. \\ &\quad \left. \left. - \min(w, kw) - H \right) \right. \\ &\quad \left. - q(n, i+k - \min(i, d), x+a + S \min(i, d) - kC - r \max(0, d-i) \right. \\ &\quad \left. - \min(w, kw) - H \right) \} \leq 0 \text{ (where } a > 0 \text{)}. \end{aligned}$$

Hence, hypothesis (ii) holds for  $n+1$ .

$$\begin{aligned} \text{iii) } q(n+1, i, x) - q(n+1, i+1, x) &\leq \\ &\max_k \left\{ \sum_{d=0}^{i-1} p(d) \left( q(n, i+k-d, x + Sd - kC - \min(w, kw) - H \right) \right. \\ &\quad \left. - q(n, i+1+k-d, x + Sd - kC - \min(w, kw) - H) \right) \\ &\quad \left. + p(i) \left( q(n, k, x + Sd - kC - \min(w, kw) - H) \right. \right. \\ &\quad \left. \left. - q(n, 1+k, x + Sd - kC - r - \min(w, kw) - H) \right) \right) \\ &\quad + \sum_{d=i+1}^M p(d) \left( q(n, k, x + Si - kC - \min(w, kw) - r(d-i) - H) \right. \\ &\quad \left. - q(n, k, x + Si + S - kC - \min(w, kw) - r(d-i) - H) \right) \} \\ &\leq \max_k \left\{ \sum_{d=0}^i p(d) \left( q(n, i+k-d, x + Sd - kC - \min(w, kw) - H) \right. \right. \\ &\quad \left. \left. - q(n, i+1+k-d, x + Sd - kC - \min(w, kw) - H) \right) \right) \\ &\quad + \sum_{d=i+1}^M p(d) \left( q(n, k, x + Si - kC - \min(w, kw) - r(d-i) - H) \right. \\ &\quad \left. - q(n, k, x + Si + S - kC - \min(w, kw) - r(d-i) - H) \right) \} \leq 0. \end{aligned}$$

Hence, hypothesis (iii) holds for  $n+1$ .



## Proof of Lemma 2

i) Suppose the optimal action in state  $(n, i + j, x - jC - w)$  is  $k(n, i + j, x - jC - w) = \delta \geq 0$ , and consider the particular action that orders  $j + \delta$  in state  $(n, i, x)$ . As  $i \geq M$ , from equation (1) we have

$$\begin{aligned}
q(n, i, x) &\geq \sum_{d=0}^M p(d) q(n-1, (j+\delta) + i - d, x + Sd - C(j+\delta) + \\
&\quad - \min(w, (j+\delta)w) - H), \text{ from Lemma 1 ii)} \\
&\geq \sum_{d=0}^M p(d) q(n-1, (i+j) + \delta - d, x - Cj - w + Sd - C\delta \\
&\quad - \min(w, \delta w) - H) \\
&= q(n, i + j, x - Cj - w)
\end{aligned}$$

as  $-\min(w, (j+\delta)w) \geq -w - \min(w, \delta w) \forall j, \delta, w \geq 0$ , arriving at the desired inequality  $q(n, i, x) \geq q(n, i + j, x - jC - w)$ .

ii) It is sufficient to prove property (ii) for  $j = 1$ , and the proof will be by induction on  $n$ . The result is trivially true for  $n = 0$  as  $q(0, i, x) = 1 \forall i \geq 0$  and  $x + \alpha Ci \geq 0$ . Assume that it is true for  $n - 1$ . Let  $I(i) = \begin{cases} 0 & \text{if } i \leq 0 \\ 1 & \text{if } i > 0 \end{cases}$ , and let  $k(n, i + 1, x) = \delta$  be the optimal action in state  $(n, i + 1, x)$ , and consider the action that orders  $\delta$  in state  $(n, i, x + S)$  for  $i > M$ , or state  $(n, i, x + S + r)$  for  $i \leq M$ . We have

$$\begin{aligned}
q(n, i + 1, x) &= \sum_{d=0}^i p(d) q(n-1, i + 1 + \delta - d, x + Sd - C\delta - \min(\delta w, w) \\
&\quad - r \max(0, d - (i + 1)) - H) + \sum_{d=i+1}^M p(d) q(n-1, \delta, x + S(i + 1) \\
&\quad - C\delta - r(d - i - 1) - \min(\delta w, w) - H) \\
&\leq \sum_{d=0}^i p(d) q(n-1, i + \delta - d, x + S + Sd - C\delta - \min(\delta w, w) \\
&\quad + rI(M - i) - H) + \sum_{d=i+1}^M p(d) q(n-1, \delta, x + S + Si \\
&\quad - C\delta - r(d - i) + rI(M - i) - \min(\delta w, w) - H) \\
&\leq q(n, i, x + S) \text{ if } i \geq M, \text{ and is} \\
&\leq q(n, i, x + S + r) \text{ if } i < M
\end{aligned}$$

where the second inequality follows from the induction hypothesis, and Lemma 1 ii). This proves the induction hypothesis holds for  $n$ , and the result follows.

### Proof of Lemma 3

$q(n, i, x)$  is bounded above by 1 and below by 0, and from Lemma 1 i) is monotonic non-increasing in  $n$ . As bounded monotonic sequences converge, point i) follows. Points ii) and iii) follow immediately by taking the limit in the results of Lemma 2, and points iv) and v) follow by taking the limit in the results of Lemma 1 parts ii) iii).

### Proof of Lemma 4

The proof uses the policy iteration algorithm for dynamic programming models (see Puterman<sup>6</sup>). Let us first focus on the parsimonious policy described in equation (4) for  $w > 0$ . When this policy is applied, the average reward and bias terms, equation (2), satisfy the following equation.

$$g + v(i) = \begin{cases} \sum_{d=0}^M p(d)(Sd - H + v(i - d)) & i \geq 2M \\ \sum_{d=0}^M p(d)(Sd - NC - w - H + v(i + N - d)) & M \leq i < 2M \\ \sum_{d=0}^M p(d)(S \min(i, d) - NC - r \max(0, d - i) - w - H + v(i + N - \min(i, d))) & i < M \end{cases},$$

It is easy to verify by substitution that the values of  $g$  and  $v(i)$  in equations (3) satisfy this equation.

Note that if we describe the two expression for  $v(i)$  in equation (3) as  $v_1(i)$  if  $i \geq M$ , and  $v_2(i)$  if  $i < M$ , then expression  $v_1(i) \geq v_2(i)$  since

$$(S - C) \sum_{d=0}^i (i - d)p(d) > (S - C)(i - \bar{d}).$$

Now apply a policy improvement step to verify that the policy of equation (4) is optimal.

For states  $i \geq 2M$ , and  $w > 0$  the policy improvement step looks for the action  $k$  which maximises

$$\sum_{d=0}^M (Sd - kC - \min(w, kw) - H + v(i + k - d))p(d),$$

and we wish to show that this occurs when  $k = 0$ . This would be the case if

$$Sd - H + v(i - d) \geq Sd - kC - w - H + v(i + k - d) \text{ for all } k.$$

Since  $i - d$  and  $i + k - d \geq M$  the corresponding bias term  $v(i - d)$ , and  $v(i + k - d)$  are in fact  $v_1(i - d)$  and  $v_1(i + k - d)$  so we require  $Sd - H + C(i - d) + (S - C)\bar{d} + (i - d)\frac{w}{N} \geq Sd - H + C(i + k - d) + (S - C)\bar{d} + (i + k - d)\frac{w}{N}$ . Simplifying we need  $(i - d)\frac{w}{N} \geq (i + k - d)\frac{w}{N} - w$ . Since  $k < N$  this inequality holds.

For  $M \leq i \leq 2M$ , we want to show that

$$\sum_{d=0}^M (Sd - NC - w - H - v(i + N - d))p(d) \geq \sum_{d=0}^M (Sd - kC - \min(w, kw) - H + v(i + k - d))p(d)$$

for any  $k$ ,  $0 \leq k \leq N$ . If  $i + k - d \geq M$  the corresponding bias term ( $v$  function) is  $v_1$ . This means we want

$$\begin{aligned} S\bar{d} - NC - w - H + C(i + N - \bar{d}) + (S - C)\bar{d} + \frac{w}{N}(i + N - \bar{d}) &\geq \\ S\bar{d} - kC - w - H + C(i + k - \bar{d}) + (S - C)\bar{d} + \frac{w}{N}(i + k - \bar{d}). \end{aligned}$$

This is trivial as  $k \leq N$ . On the other hand if  $i + k - d < M$ , then the right hand side is  $v_2(i + k - d)$  not  $v_1(i + k - d)$  and since  $v_2(i + k - d) \leq v_1(i + k - d)$  the inequality still holds.

For  $i \leq M$ , we want to show that

$$\begin{aligned} \sum_{d=0}^M p(d)(S \min(i, d) - NC - w - H - r \max(0, d - i) + v_1(i + N - d)) &\geq \\ \sum_{d=0}^M p(d)(S \min(i, d) - kC - r \max(0, d - i) - w - H + v(i + k - \min(i, d))) & \end{aligned}$$

This reduces to the previous case since the shortage  $r$ , and the sales  $S$  terms cancel on both sides and  $v(i + k - \min(i, d)) \leq v_1(i + k - \min(i, d))$ .

Now look at the parsimonious policy for  $w = 0$ , and  $i \leq 2M$ . First evaluate the policy described by equation (5). When the policy is applied, the average reward and bias terms satisfy the following equation.

$$g + v(i) = \begin{cases} \sum_{d=0}^M p(d) & (Sd - (2M - i)C - H + v(2M - d)) & \text{for } i > M \\ \sum_{d=0}^M p(d) & (S \min(i, d) - MC - r \min(0, d - i) - H \\ & + v(i + M - \min(i, d))) & \text{for } i \leq M \end{cases}$$

It is easy to verify by substitution that the values of  $g$  and  $v(i)$  in equations (3) satisfy this equation.

Now apply a policy improvement step to verify that the policy is optimal. For  $i > M$  the policy improvement step looks for the action  $k$  which maximises

$$\sum_{d=0}^M (Sd - kC - H + v(i + k - d))p(d).$$

Since

$$v(i + 1) - v(i) = S \sum_{d=i+1}^M p(d) + C \sum_{d=0}^i p(d) + r \sum_{d=i+1}^M p(d) > C \text{ if } i < M,$$

and  $v(i + 1) - v(i) = C$  if  $i \geq M$ , this expression is maximised when  $k$  is chosen so that  $i + k - d \geq M$  for all possible values of demand,  $d$ . Hence any  $k \geq 2M - i$  is optimal.

For  $i \leq M$  the optimal policy improvement step looks for the action  $k$  that maximises

$$\sum_{d=0}^i p(d)(Sd - kC - H + v(i + k - d)) + \sum_{d=i+1}^M p(d)(Si - kC - r(d - i) - H + v(k))$$

Since  $v(i + 1) - v(i) > C$  if  $i < M$ , and  $v(i + 1) - v(i) = C$  if  $i \geq M$ , this expression is maximised when  $k$  is chosen so that  $k \geq M$ . Hence, the policy given by equation (5) is optimal.

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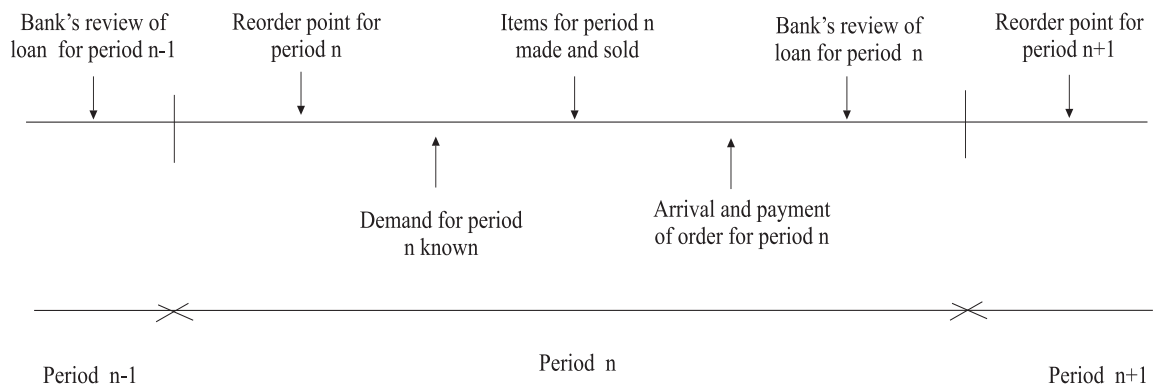
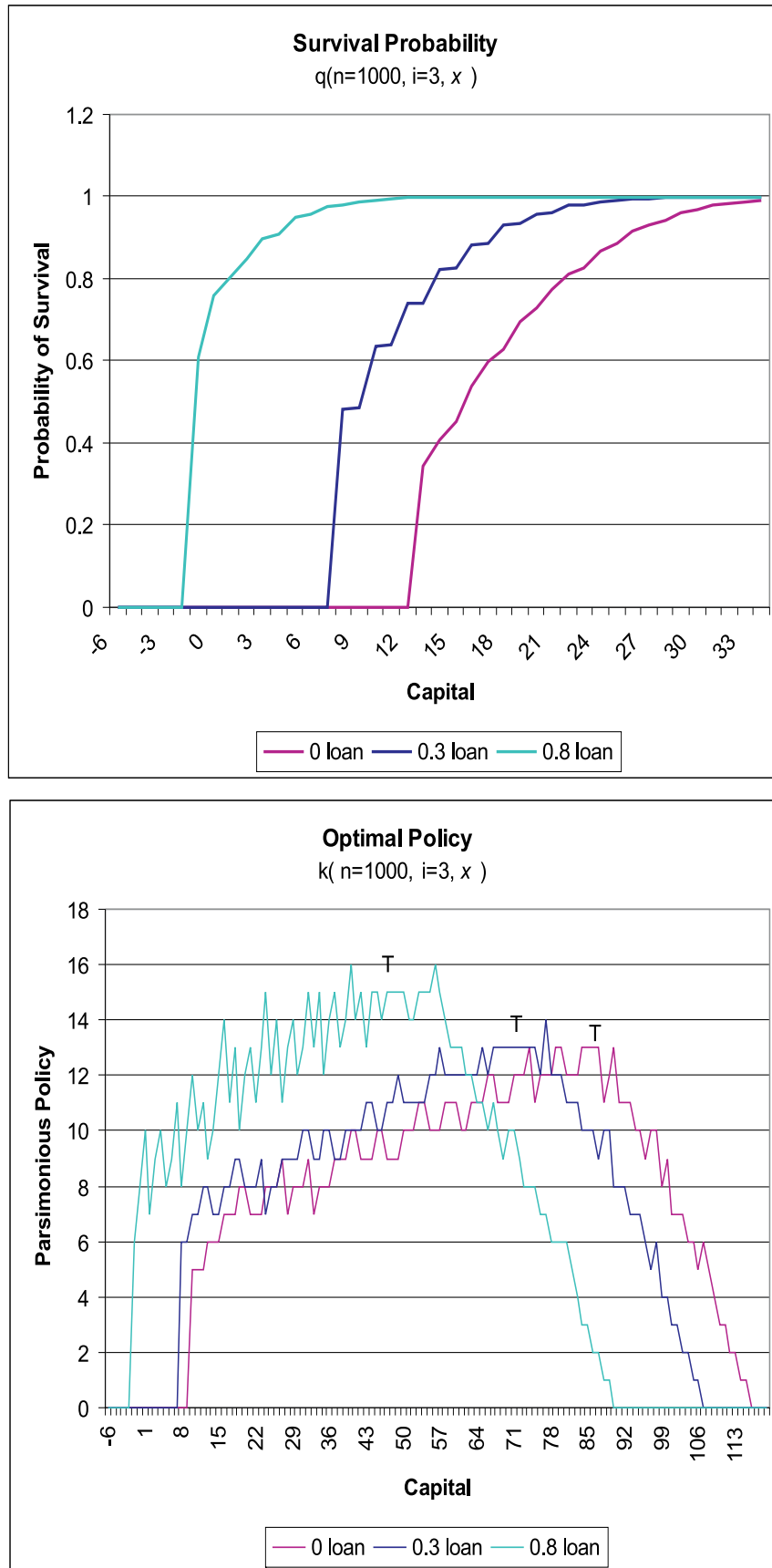


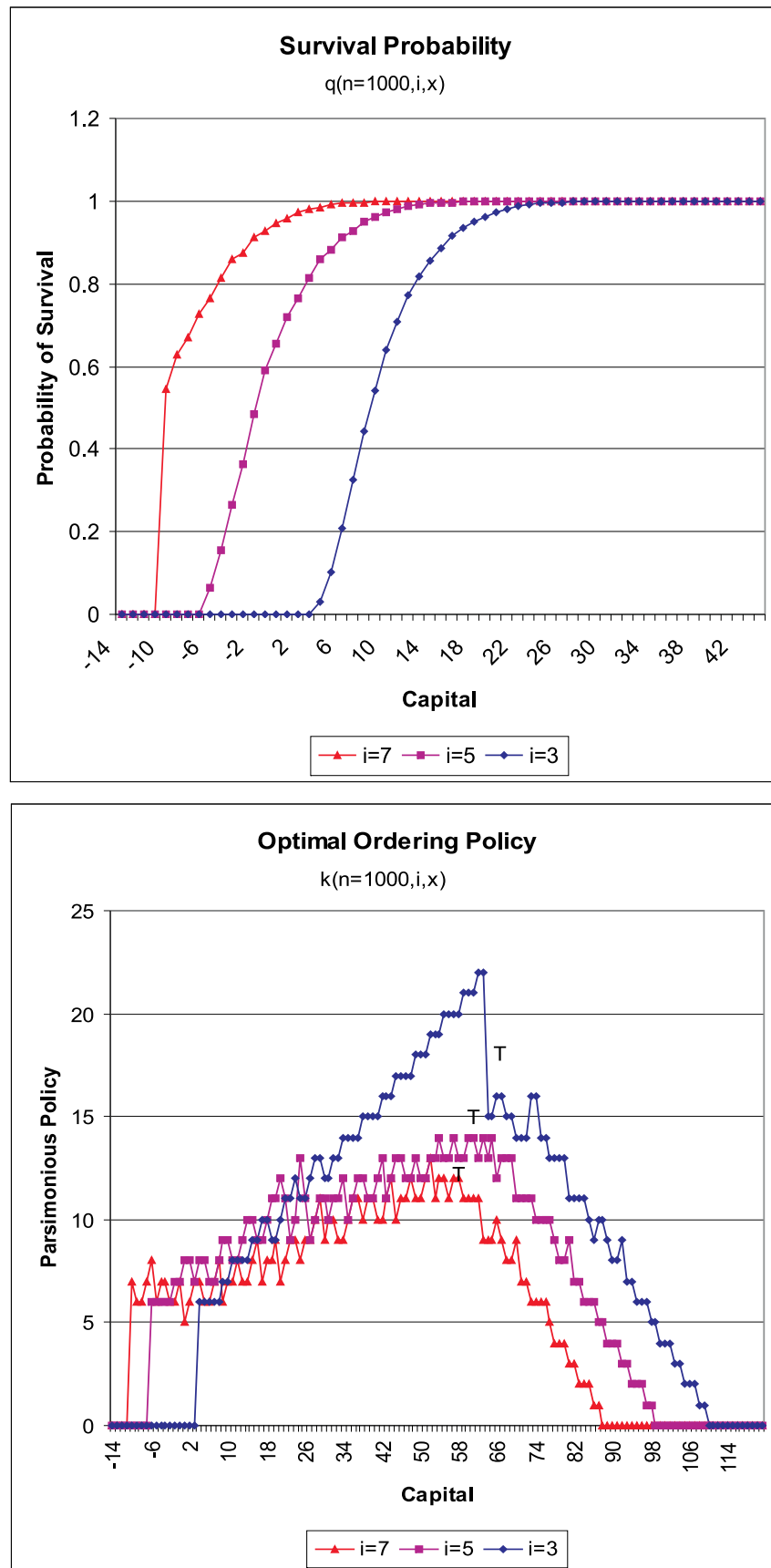
Figure 1: A time line for the events in one period

Figure 2: Survival probability and Optimal Policy for different  $\alpha$  loans



T=point at where there starts to be more than one optimal policy

Figure 3: Survival probability and ordering policies for different inventory levels.



T=point at where there starts to be more than one optimal policy



Figure 4: Survival probability and ordering policies for different ordering costs.

